Term Paper: A Literature Review of Fair Sharing and Envy-freeness

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Abstract

Envy-freeness is an important notion to represent fairness in item allocating for n people. In this term paper, we review notions of envy-freeness, issue known properties about the notions, and consider their existence known till now from these papers. Moreover, we also review the results of the efficiency issue in fair sharing.

Envy-freeness

The problem of fair division is first mathematically discussed by Steinhaus in the paper (Steinhaus 1949). Here we formalize the problem of fair sharing for continuous cases in the following definition:

Definition. A **(continuous) fair sharing instance** is a triple $([n], C, V)$, where we have:

- $[n]$: the set of n agents.
- $C = [0, 1]$: the item to divide, or to say, "the cake".
- V is a set of n valuations $V_1, ..., V_n$, V_i is a function from any subset of the cake to $\mathbb{R}^+$. (Here we consider the case when agents are hungry and any intervals of non-zero length gives a positive value.)

This is the problem of allocating "the cake". And now we consider how to measure fairness in a division. Intuitively, we can think fairness as everybody thinks his part is the largest and doesn't envy other people. This is the definition of envy-free. And we can also regard fairness as every person gets at least $1/n$ in his valuation, this is called proportional. Formally, the definition is:

Definition. An allocation X is an **envy-free** allocation in the fair sharing instance if for each agent i, he regards $V_i(X_i) \geq V_i(X_j)$ for all $j \neq i$.

Definition. An allocation X is **proportional** in the fair sharing instance if for each agent i, he regards $V_i(X_i) \geq V_i(C)/n$.

It's easy to see that any envy-free allocation is proportional. The classical and simple case when $n = 2$, where two people divide a cake, can be shown to have an envy-free allocation. The procedure is done by:

- First, one person divide the cake equally in his valuation.
- Then, the other person chooses the more valuable part in his view.

In this allocation procedure, we can see no one may envy the other one. Furthermore, for the general case, (Brams and Taylor 1995) first proved that there's an envy-free allocation for all n by giving a protocol for $n = 3$ and general n to extend the proof of the $n = 2$ case. Then (Edward Su 1999) gives a more simple proof for the existence of an envy-free division using Sperner's lemma. Here we sketch the proof:

Theorem 1. There's an envy-free allocation in the above continuous case.

Proof(Edward Su 1999). Consider dividing the cake into n intervals by giving $(n-1)$ cuts from left to right. Consider the actual length being $x_1, ..., x_n$, then $x_i \geq 0$ and $\sum_i x_i = 1$. Therefore, the possible allocations of this type constitute a simplex in the n-dimensional euclidean space.

Now we color every node in the following logic:

- If we determine to let agent i to choose at a node, with the node's partition, and he chooses the interval j, then use color j to color the node.
- For any small simplex we consider, we will determine to let all n agents to choose once corresponding to one of its nodes.

Then if the simplex is small enough, we can surely give an envy-free allocation. Originally, we have n nodes, and on any node e_i, the ith interval is the whole cake and it must be colored i. On any face with $x_i = 0$, nobody will choose i since it is with zero value and there's some with more. Therefore, the coloring is a Sperner coloring.

After that, the paper considered to divide the simplex using barycentric subdivision given by (Vick 1994). For $n = 2$ cases, the division is just iteratively connecting any node of the triangle to the barycenters of its opposite edge and divide it into 6 triangles, and for general n the case is more complex. Finally, the theorem is proven by Sperner's lemma.

Moreover, the paper (Edward Su 1999) also solves the rental harmony problem using Sperner's lemma.

Therefore, the original existence problem of fair allocation problem is settled. Some remain problems are about its existence in extensions. (Segal-Halevi 2017) removed the assumption that consider cases which the value of a part can be positive, negative or zero and prove the result for $n = 3$ and (Meunier and Zerbib 2019) and (Avvakumov and Karasev 2021) gave more general results for the problem with the boundary condition removed. (Hosseini, Igarashi, and Searns 2020) and (Igarashi and Meunier 2021) considered dividing multi-layered cakes with n agents and m intervals.

Another issue is the computational complexity in finding such a division for cakes. Here, (Stromquist 2008) shows that no finite protocol can be used to compute the exact envy-free cake division with general cases for $n \geq 3$ where every person get a continuous part. (Deng, Qi, and Saberi 2012) further showed that it is as hard as finding a Brouwer's fixed point, but gave a FPTAS for $n = 3$ case with monotone utility. (Cohler et al. 2011) and (Aziz and Mackenzie 2016) gave cake-cutting algorithms in several different cases.

Another interesting fact is, our proof for $n = 2$ shows a protocol knowing only one person's valuation (and let the other choose). (Woodall 1980) showed that a envy-free protocol can be obtained even if one agent's valuation is not known for general n, and (Meunier and Su 2019) gave more general result for Sperner's lemma and Fan's lemma and showed there's a division into $(n-1)$ parts which can be envy free with anyone absent and others choose their parts.

Envy-freeness up to one item (EF1)

Although envy-freeness for continuous cases can be obtained in theory, in discrete cases, it may be another story. Let's consider the easiest case when two agents are to allocate one discrete item, then there's no envy-freeness since the one without any item must be envious of the other. In this case, to represent fairness in sharing, we need to relax the requirement of envy-freeness. We first formalize the discrete fair sharing instance by:

Definition. A **(discrete) fair sharing instance** is a triple $([n], M, V)$, where we have:

- $[n]$: the set of n agents.
- M: the m items to divide.
- V is a set of n valuations $V_1, ..., V_n$, $V_i : 2^M \to \mathbb{R}^+$ is a function from any subset of M to $\mathbb{R}^+$. The valuation is additive if $V_i(X) = \sum_{g \in X} V_i(\{g\})$, that is, the valuation of a set is the sum of the valuations of its items.

Then (Budish 2011) gives the definition of EF1, which is a relaxation of envy-freeness by considering to delete one item of j to make i no more envy, which gives the definition that for all i, j, there exists an item g of j, such that after deleting g, i no more envy j. Formally:

Definition(Budish 2011). An allocation X is **envy-free up to one item (EF1)** if for all i and j, there exists an item $g \in X_j$ such that $V_i(X_i) \geq V_i(X_j - \{g\})$.

Remark that, we can surely define EFk for all number of items k, however, in this case the definition is not important since we know that EF1 allocations exist. This result is first shown by (Lipton et al. 2004). This paper gave a poly-time algorithm for finding an EF1 allocation using the definition of envy-cycles. Here we sketch the proof:

Theorem 2. There's an $O(mn^3)$ algorithm to find an EF1 allocation.

Proof(Lipton et al. 2004). First we define something used in our proof:

- The envy of p to q is $e_{ij}(X) = max(0, V_i(X_j) - V_i(X_i))$, and the envy of the whole allocation is $e(X) = max_{i,j} e_{ij}(X)$.
- The envy graph is with n nodes corresponding to agents, and if i envies j ($V_i(X_j) > V_i(X_i)$), there's an edge from i to j.

Lemma. For any partial allocation A, we can find a new allocation B with acyclic envy graph and $e(B) \leq e(A)$.

The proof of lemma is to notice envy cycles in the graph. If we find a cycle, then we can do a cyclic shift to let everyone obtain the set he envies on the cycle. Then everyone on the cycle can get a better utility. Then we can see the envy in the new partial allocation is less. For agents in the cycle, the edges for him to envy decreases by at least 1, while other agents has the same amount of envy edges out. So iteratively doing this can get an acyclic graph we desire.

Now the algorithm does:

- On every turn j, give item j to an agent without in-degree in the envy graph.
- Construct the new envy graph of A and use lemma to get an allocation B with acyclic envy graph and $e(B) \leq e(A)$.

We can see this algorithm gives an EF1 allocation by induction since we can verify that with adding items, that item can be deleted and then nobody envies, and generate B from A certainly doesn't gviolate the requirement of EF1. The algorithm can be proved to run in $O(mn^3)$ time. Therefore, this existance problem is solved. It's remarkable that, the paper also gave a FPTAS for the minimum envy problem with additive valuations.

Now we consider an alternative idea with efficiency also considered in the work by (Varian 1973) (with cake cutting) and (Caragiannis et al. 2019) (with discrete allocating). We use Pareto optimality and Nash welfare to measure the allocation's efficiency here. We have:

Definition(Arrow and Intriligator 2000). The **Nash welfare(NW)** of an allocation X in the fair sharing instance is the product of every agent's utility, that is, $\prod_i V_i(X_i)$.

Definition. An allocation X is **Pareto optimal(PO)** if there's no X' such that for all i, $V_i(X'_i) \geq V_i(X_i)$ and there exists i with $V_i(X'_i) > V_i(X_i)$.

It is easy to show that the allocation maximizing Nash welfare is PO, and (Caragiannis et al. 2019) showed more:

Theorem 3. A maximum Nash welfare (MNW) allocation is both PO and EF1 in positive additive valuations.

Proof(Caragiannis et al. 2019). We only consider the case that everyone has some utility since else we can allocate a subset of people one item each to guarantee the EF1 property.

Now consider an allocation which is MNW but not EF1. If i envies j even if after deleting any item in X_j, then we consider giving something from j to i. Let g^* be the item in X_j minimizing $V_j(g)/V_i(g)$. Then after giving g^* from j to i we have:

$$\frac{NW(X')}{NW(X)} > 1 \Leftrightarrow \frac{V_j(g^*)}{V_i(g^*)}(V_i(X_i) + V_i(g^*)) < V_j(X_j)$$

This can be obtained from the minimization:

$$\frac{V_j(g^*)}{V_i(g^*)} \leq \frac{V_j(X_j)}{V_i(X_j)}$$

With the fact that EF1 is violated:

$$V_i(X_i) + V_i(g) < V_i(X_j)$$

Therefore, we know a MNW allocation is both EF1 and PO, which means it has good properties considering both fairness and efficiency.

It is shown (Nguyen, Roos, and Rothe 2013) that MNW allocation is NP-hard. However, the paper gave an algorithm running fast in common cases on *spliddit.org* and giving possibility that this division can sometimes be used in reality.

After the existence of EF1 allocation is settled, some papers also focused on extensions and computation issues. We can see that this property is mature in some degree with lots of extensions. For example, (Bilò et al. 2018) considered the EF1 allocation problem for items on a graph and one must get connected pieces. (Barman et al. 2019) considered the complexity of EF1 property in truthful auctions. (Biswas and Barman 2018) considered the existence of EF1 in cardinality and matroid constraints. (Aziz et al. 2019) gave new algorithms for EF1 and discuss the EF1 and PO properties when there's negative utility. In general, more and more extensions are caring about more complex cases, and discuss EF1 property in these cases.

Finally we remark that, there's another relaxation of proportionality, which is called maximin share (MMS) also by (Budish 2011). It is proved by (Procaccia and Wang 2014) that MMS allocations possibly not exists in some allocation instances. And there are multiple works now to get approximate MMS sharings.

Envy-freeness up to any item (EFX)

We noticed that EF1 shows that "envy disappears after the removal of the most valuable item", however, that item "might be the primary reason for very large envy to exist" (Chaudhury, Garg, and Mehlhorn 2020). Therefore, we need a stronger notion than EF1, which is also a relaxation of EF, to describe the fairness in item allocating. Here, (Caragiannis et al. 2019) gave the definition of envy-freeness up to any item (EFX), which allows envy to disappear even after the removal of the least valuable item. We can define EFX in disctete problems as:

Definition(Caragiannis et al. 2019). An allocation X is **envy-free up to any item (EFX)** if for all i and j, for all item $g \in X_j$, we have $V_i(X_i) \geq V_i(X_j - \{g\})$.

EFX may be the most convincible definition of fairness we know till now, however, EFX allocations are also obviously harder to get compared to EF1 with higher requirements. We can see that the techniques we discussed in EF1, which are envy cycles and MNEs, cannot be directly used to get an allocation with EFX. Therefore, the related research is focused on the existance of EFX. Several papers have shown that EFX allocation exists in special cases, however, the general problem questioning EFX allocation exists or not is still open. And this problem may be the most important one now in this field.

Let's first consider the easiest case when the valuations of agents are identical. In this case, EFX allocations are proved to exist by (Plaut and Roughgarden 2020). Here we consider the same valuation as V and we have a theorem:

Theorem 4. EFX allocation exists in the case with positive additive valuations with everyone identical.

Proof(Plaut and Roughgarden 2020). We consider such a procedure: every time if the set $X_j - g$ is envied by somebody, we give g from j to one with minimal utility, say i, then we know the utility of i strictly increases (since we give positive valuation to him) and the utility of j is also strictly larger than the original utility of i (since j is envied).

Therefore, the minimal utility strictly increases or the number of person with that utility decreases by 1. So we know the procedure will end. Then we get an EFX allocation by this procedure.

This theorem at least gives us a point to start, however, even for some simple cases, the proof known by us may be highly nontrivial. Therefore, probably we can't easily extend any known proof to the general case with low effort.

EFX is known to be true in various constraints. (Mahara 2020) shows that if there's only two possible valuations in all agents, there's an EFX allocation (hence for the two person case). (Chaudhury, Garg, and Mehlhorn 2020) gives a proof for the three person case with additive valuations.

One relaxation of EFX is to get approximate results. The first result is done by (Plaut and Roughgarden 2020), which got an $1/2$-EFX allocation. Here, an α-EFX allocation means $V_i(X_i) \geq \alpha V_i(X_j - \{g\})$ for all i, j and item $g \in X_j$. (Amanatidis, Markakis, and Ntokos 2020) beats the $1/2$ value to 0.618 and (Chaudhury et al. 2021) gives proof to all small constants ϵ.

Another relaxation of EFX, which is called "EFX with charity", allowing giving a subset an EFX allocation with high social welfare, is proven by (Chaudhury et al. 2020) (there's a trivial case when we allocate an empty subset, but the article proved that it can be more efficient).

Efficiency issues are also considered. (Plaut and Roughgarden 2020) shows EFX and PO may not coexist in an allocation, this is different from EF1, and (Amanatidis et al. 2021) gives answer to whether a MNW allocation can be EFX in situations. (Chaudhury, Garg, and Mehlhorn 2020) shows that there's cases with partial EFX allocation not pareto dominated by any full EFX allocation and even with higher Nash welfare. (Schulman and Vazirani 2012) and (Hosseini, Igarashi, and Searns 2020) considered EFX allocations with efficiency in lexicographic preference.

Summary

We mainly talk about three fairness issues: EF, EF1 and EFX with envy-freeness. For EF in continuous cases and EF1 in discrete cases (even with PO), the existence are clear and the recent research focused on more extensions like computation complexity, more complex conditions and cases, or the applications into real world's division. For EFX, the definition is reasonable but not known to have a proof for existence. Therefore, recent papers focus on settling this open problem (or at least in easy cases), and some papers also consider further.

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